Solving a System of Equations

Objectives

- Understand how to solve a system of equations with:
  - Gauss Elimination Method
  - LU Decomposition Method
  - Gauss-Seidel Method
  - Jacobi Method

A system of linear algebraic equations

Linear algebraic equations are of the general form,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

Where the a's are constant coefficients, the b's are constants, the x's are unknowns, and n is the number of equation. The algebraic equations can be briefly rewritten in the matrix form as follows:

$$[A]\{x\} = \{b\} \tag{2}$$

Gauss Elimination Method

There are many techniques for solving a system of linear algebraic equations. One of basic techniques that can be applied to large sets of equation and can be formalized and programmed for the computer is **Gauss Elimination Method.**

The procedure consists of two major steps which are

1. Forward Elimination – the equations are manipulated to eliminate the unknowns from the equations until we have one equation with one unknown.
2. Back substitution – The equation with one unknown can be solved directly. The result is back-substituted into one of the original equations to solve for the remaining unknown.

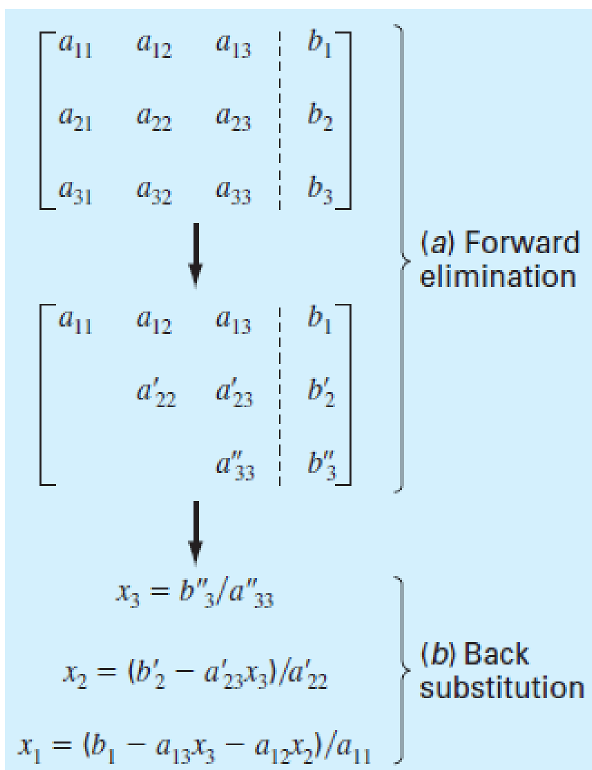

Fig. 1 The steps of Gauss Elimination

<u>Example 1</u> Use Gauss Elimination to solve

$$2x_1 + x_2 - 5x_3 = -21$$
$$x_1 + 2x_2 - 2x_3 = -15$$
$$x_1 - 4x_2 + x_3 = 18$$

Example 2 Use Gauss Elimination to solve

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\begin{aligned}
2x_2 + x_3 \quad &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 - 3x_2 + 6x_3 &= 18
\end{aligned}
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<u>Example 3</u> Use Gauss Elimination to solve

$$0.0003x_1 + 3.0000x_2 = 2.0001$$
$$1.0000x_1 + 1.0000x_2 = 1.0000$$

Exact Solutions:
$$x_1 = 1/3 = 0.3333$$
$$x_2 = 2/3 = 0.6667$$

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	−3.33	1099
4	0.6667	0.0000	100
5	0.66667	0.30000	10
6	0.666667	0.330000	1
7	0.6666667	0.3330000	0.1

If we rearrange the pivoting equation by setting the largest element as the pivot element.

$$1.0000x_1 + 1.0000x_2 = 1.0000$$
$$0.0003x_1 + 3.0000x_2 = 2.0001$$

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	0.333	0.1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001
6	0.666667	0.333333	0.0001
7	0.6666667	0.3333333	0.0000

LU Decomposition Method

The forward elimination of Gauss elimination comprises the bulk of the computational effort, especially the large system of equations. LU decomposition method separates the time consuming elimination of the matrix [*A*] by the following steps.

1. **LU decomposition step** – The matrix [*A*] is decomposed into lower triangular matrix [*L*], and upper triangular matrix [*U*].
2. **Substitution step** – [*L*] and [*U*] are used to determine a solution {*x*} for a right-hand side {*b*}. This step consists of the forward and back substitutions.
   2.1 **The forward substitution** is conducted to determine the intermediate vector {*d*} from [*L*]{*d*}={*b*}.
   2.2 Then, the result of {*d*} is substituted into [*U*]{*x*}={*d*}, and solve for {*x*} through **the back substitution.**

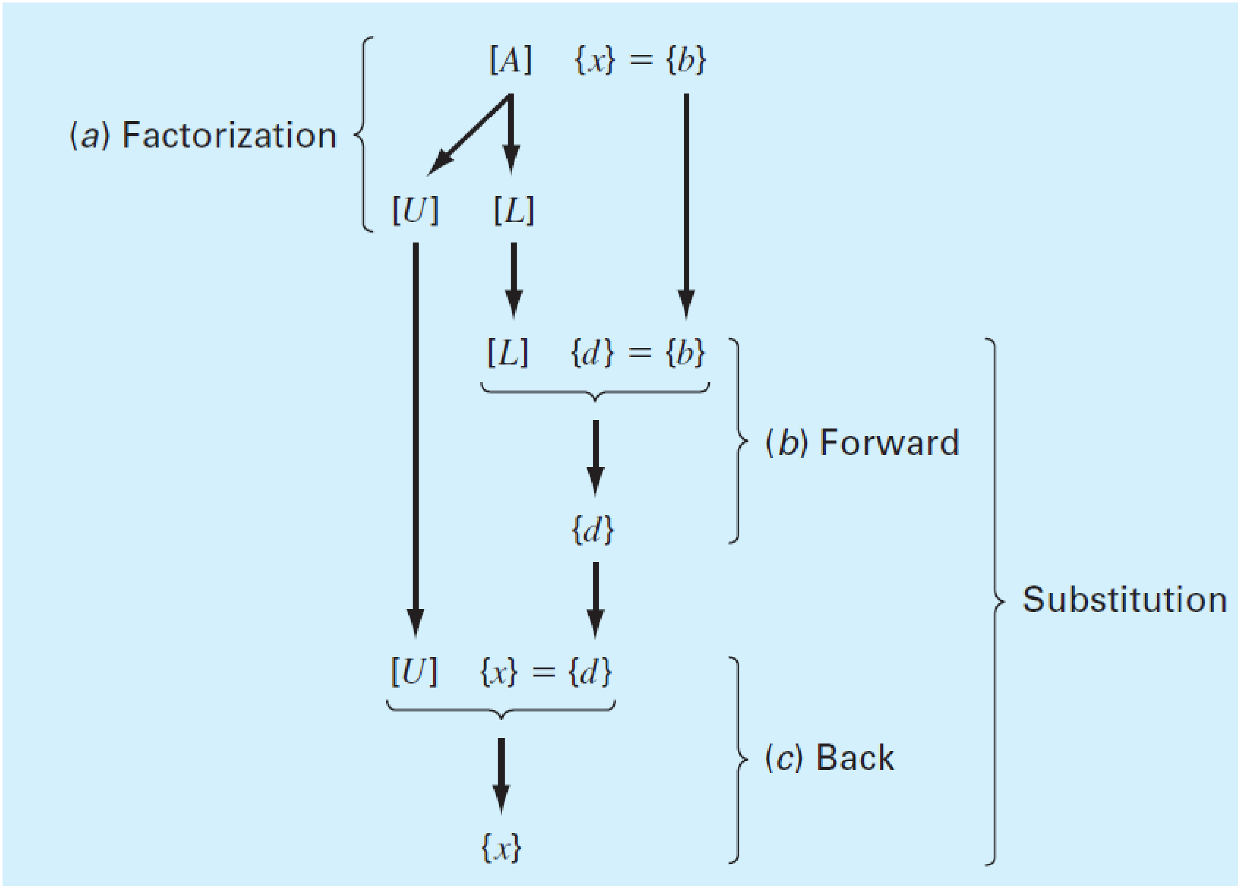

Fig.2 The steps in LU decomposition.

1. <u>The LU decomposition/ Factorization step</u>

$$[A]=[L][U]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$u_{11}=a_{11} \qquad u_{12}=a_{12} \qquad u_{13}=a_{13}$$

$$l_{21}u_{11}=a_{21} \qquad l_{21}u_{12}+u_{22}=a_{22} \qquad l_{21}u_{13}+u_{23}=a_{23}$$

$$l_{31}u_{11}=a_{31} \qquad l_{31}u_{12}+l_{32}u_{22}=a_{32} \qquad l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{32}$$

[L] and [U] can be obtained by solving the above 9 equations.

2. The substitution step

We start from

$$[A]\{x\} = \{b\} \tag{1}$$

After LU decomposition step, we have

$$[L][U]\{x\} = \{b\} \tag{2}$$

Now, let

$$[U]\{x\} = \{d\} \tag{3}$$

Eq. (2) can be rewritten as

$$[L]\{d\} = \{b\} \tag{4}$$

Or

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{5}$$

Then, determine {*d*} through forward substitution. After we get {*d*}, we substitute {*d*} into eq. (3). We have

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{6}$$

Use back substitution to determine {*x*}.

Example 4 Use LU decomposition to solve

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

LU decomposition/Factorization

$$[A] = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.1481 & 1 \end{bmatrix} \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3519 \end{bmatrix}$$

Substitution step for determine {*d*} and {*x*}

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Iterative Methods for Systems of Equations

- Jacobi Method
- Gauss-seidel Method

Jacobi Method

Assume that we have a 3x3 set of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Or

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (1)$$

If the diagonal elements are all nonzero, the first equation can be solved for x_1, the second for x_2, the third for x_3. Then, we have

$$x_1^j = \frac{b_1 - a_{12}x_2^{j-1} - a_{13}x_3^{j-1}}{a_{11}} \qquad (2a)$$

$$x_2^j = \frac{b_2 - a_{21}x_1^{j-1} - a_{23}x_3^{j-1}}{a_{22}} \qquad (2b)$$

$$x_3^j = \frac{b_3 - a_{31}x_1^{j-1} - a_{32}x_2^{j-1}}{a_{33}} \qquad (2c)$$

where *j* and *j-1* are the present and previous iterations. To solve the solutions, we need the initial guesses for x_1^j, x_2^j, x_3^j for the first iteration. Then, the iteration is continued until our solutions converge closely enough to the true values or the error of the approximation less than or equal to tolerance.

$$\varepsilon_{a,i} = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| \times 100 \quad \le \varepsilon_s \tag{3}$$

where *i* = 1, 2, 3 for the x's.

Example 5 Use Jacobi method to solve

$$\begin{aligned} 2x_1 + x_2 - 5x_3 &= -21 \\ x_1 + 2x_2 - 2x_3 &= -15 \\ x_1 - 4x_2 + x_3 &= 18 \end{aligned}$$

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Gauss-seidel Method

This method is the most commonly used iterative method for solving linear algebraic equations. Assume that we have a 3x3 set of equations.

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\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\tag{1}
$$

If the diagonal elements are all nonzero, the first equation can be solved for x_1, the second for x_2, the third for x_3. Then, we have

$$x_1^j = \frac{b_1 - a_{12}x_2^{j-1} - a_{13}x_3^{j-1}}{a_{11}} \tag{2a}$$

$$x_2^j = \frac{b_2 - a_{21}x_1^{j} - a_{23}x_3^{j-1}}{a_{22}} \tag{2b}$$

$$x_3^j = \frac{b_3 - a_{31}x_1^{j} - a_{32}x_2^{j}}{a_{33}} \tag{2c}$$

where *j* and *j-1* are the present and previous iterations. To solve the solutions, we need the initial guesses for x_1^j, x_2^j, x_3^j for the first iteration. Then, the iteration is continued until our solutions converge closely enough to the true values or the error of the approximation less than or equal to tolerance.

$$\varepsilon_{a,i} = \left| \frac{x_i^j - x_i^{j-1}}{x_i^j} \right| \times 100 \quad \leq \varepsilon_s \tag{3}$$

where i = 1, 2, 3 for the x's.

Example 6 Use Gauss-Seidel method to solve

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

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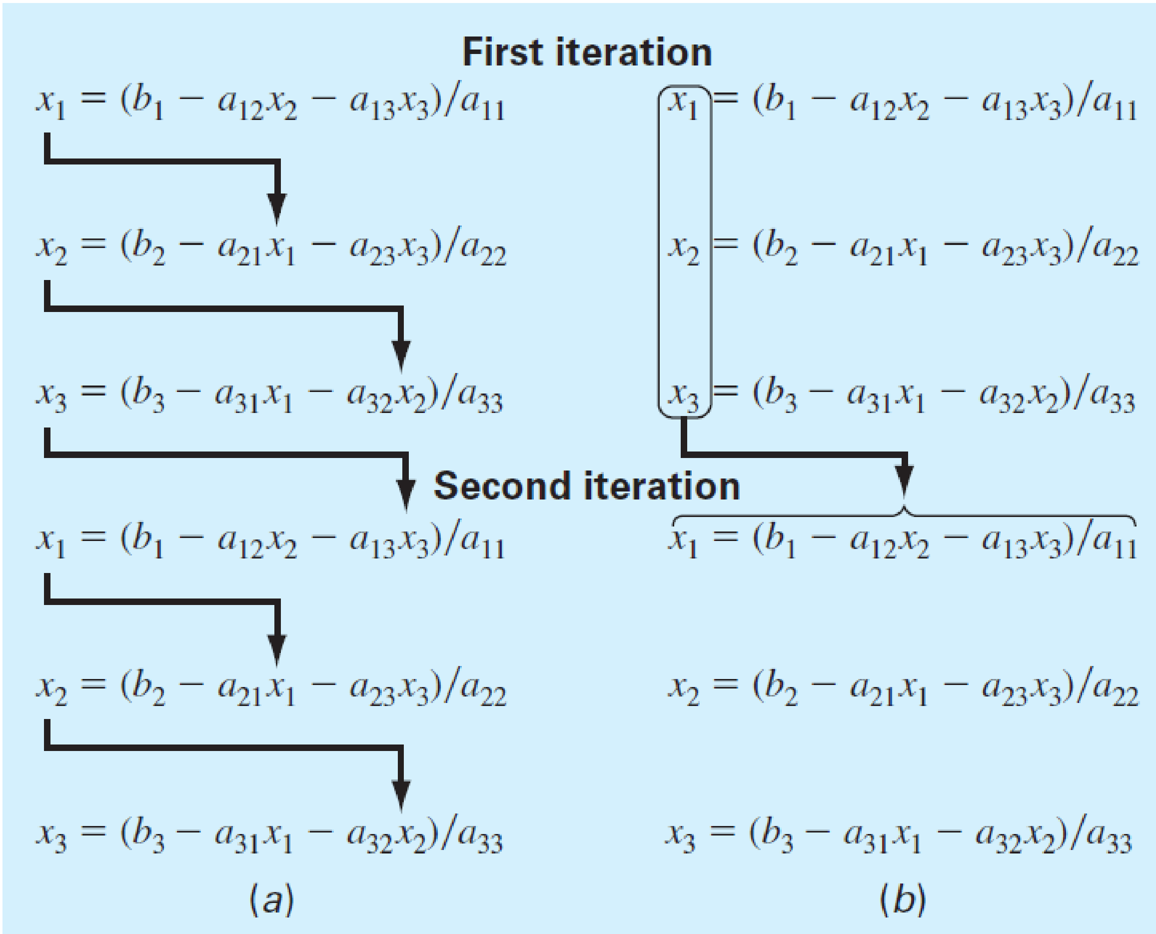

Fig. 3 The difference procedure between (a) the Gauss-Seidel method and (b) the Jacobi method.

<u>Exercise</u>

1. $$\begin{aligned} 10 &= 3x_2 - 7x_1 \\ 4x_2 + 7x_3 + 30 &= 0 \\ x_1 - 7x_3 &= 40 - 3x_2 + 5x_1 \end{aligned}$$

   Solve the above system of equations using Gauss elimination, Jacobi, and Gauss-Seidel methods. Use three iterations for the Jacobi, and Gauss-Seidel iterations.

2. The position of three masses suspended vertically by series of identical springs can be modeled by the following steady-state force balances:

$$0 = k(x_2 - x_1) + m_1 g - kx_1$$
$$0 = k(x_3 - x_2) + m_2 g - k(x_2 - x_1)$$
$$0 = m_3 g - k(x_3 - x_2)$$

If g = 9.81 m/s2, m1 = 2 kg, m2 = 3 kg, m3 = 2.5 kg, and the k's = 10 N/m. Use the Gauss elimination, Jacobi, and Gauss-Seidel methods to solve for position (the x's) of masses. Note that three iterations are performed for the Jacobi, and Gauss-Seidel iterations.

3. Use the Jacobi, and Gauss-Seidel methods to solve the following system until the percent relative error falls below $\varepsilon_s = 5\%$.

$$\begin{bmatrix} 0.8 & -0.4 & \\ -0.4 & 0.8 & -0.4 \\ & -0.4 & 0.8 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 41 \\ 25 \\ 105 \end{Bmatrix}$$